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EDGE DOMINATION IN GRAPHS OF CUBES

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To the memory of Ivan Havel

Abstract. The signed edge domination number and the signed total edge domination number of a graph are considered; they are variants of the domination number and the total domination number. Some upper bounds for them are found in the case of the n-dimensional cube Q_n.

Keywords: signed edge domination number, signed total edge domination number, graph of the cube of dimension n

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In this paper we shall treat three numerical invariants of undirected graphs which concern edge domination. We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$, its edge set by $F(G)$.

A subset D of $E(G)$ is called edge dominating in G, if each edge of G either is in D, or is adjacent to an edge of D. (Two edges are adjacent, if they have an end vertex in common.) The minimum number of edges of an edge dominating set in G is called the edge domination number [5] of G and is denoted by $\gamma'(G)$.

In [7], B. Xu introduced the signed edge domination number of G, as an analogue of the signed domination number [1]. A similar numerical invariant, the signed total edge domination number, was introduced in [8].

For each $e \in F(G)$ the symbol $N(e)$ denotes the open neighbourhood of e in G, i.e. the set of all edges which are adjacent to e in G. Further, $N[e] = N(e) \cup \{e\}$ is the closed neighbourhood of e in G.

If f is a mapping of $E(G)$ into a set of numbers and $S \subseteq E(G)$, then $f(S) = \sum\limits_{x \in s} f(x)$. The number $w(f) = f(E(G))$ is called the weight of the mapping f.

Let $f\colon E(G) \to \{-1,1\}$. The mapping f is called a signed edge dominating function (shortly SEDF) of G, if $f(N[e]) \geqslant 1$ for each $e \in E(G)$, and it is called a signed total edge domination function (shortly STEDF) of G, if $f(N(e)) \geqslant 1$ for each $e \in E(G)$. The minimum weight $w(f)$ of an SEDF (or STEDF) of G is called the signed edge domination number $\gamma'_s(G)$ of G (or the signed total edge domination number $\gamma'_{st}(G)$ of G, respectively).

In this paper we will study these concepts for the graphs of cubes. The graph Q_n of the n-dimensional cube is the graph whose vertex set consists of all Boolean vectors of dimension n (i.e. vectors, all of whose coordinates are in $\{0,1\}$) and in which two vertices are adjacent if and only if they differ in exactly one coordinate (see e.g. [2], [6])

In a graph Q_n, for $i = 1,\dots,n$ we denote by M_i the set of all edges of Q_n which join vertices differing in the i-th coordinate. Further, M_i^0 (or M_i^1) will denote the subset of M_i consisting of edges e such that the end vertex of e with the i-th coordinate 0 has even (or odd, respectively) sum of coordinates. Evidently $M_i^0 \cap M_i^1 = \emptyset$, $M_i^0 \cup M_i^1 = M_i$.

We shall find only upper bounds for these numerical invariants which can be done by showing the corresponding set.

Theorem 1. *For each positive integer n the following inequality holds:*

$$\gamma'(Q_n) \leqslant 2^{n-1}.$$

Proof. Evidently, for $i = 1,\dots,n$ the set M_i is an edge dominating set in Q_n and $|M_i| = 2^{n-1}$.

Note that the size of the edge dominating set in a cube with the minimum cardinality equals the size of the matching of this graph with the minimum cardinality, denoted by $m(Q_n)$. R. Forcade [3] has proved that $m(Q_n)/|V(Q_n)| \to \frac{1}{3}$ for $n \to \infty$, where $m(Q_n)$ is the same as $\gamma'(Q_n)$. His conjecture that $m(Q_n) = \lceil n \cdot 2^n/(3n-1)\rceil$ was disproved independently by J.-M. Laborde (by means of a computer) and by I. Havel and M. Křivánek [4] (without a computer, showing that $m(Q_n) \geqslant 24$).

For the study of other invariants we introduce some auxiliary concepts and lemmas.

If f is a mapping of $E(G)$ into $\{-1,1\}$ and v is a vertex of G, then $sl(G,f,v)$ denotes the sum of values $f(e)$ for all edges e of G which are incident with v. If H is an induced subgraph of G, then $s(H,f,v)$ has the same meaning, taking the restriction of f onto H.

We say that a mapping $f\colon F(G) \to \{-1,1\}$ has the property VS1 (or VS2), if for each $v \in V(G)$ we have $s(G,f,v) \geqslant 1$ (or $s(G,f,v) \geqslant 2$, respectively). $\square$

Lemma 1. *If a function* $f\colon E(G) \to \{-1,1\}$ *has the property* VS1, *then it is a* SEDF. *If it has the property* VS2, *then it is a* STEDF.

Proof. Let u, v be the end vertices of an edge e. If f has VS1, then $f(N[e]) = s(G,f,u) + s(G,f,v) - f(e) \geqslant 1+1-1 = 1$. If f has VS2, then $f(N(e)) = s(G,f,u) + s(G,f,v) - 2f(e) \geqslant 2+2-2 = 2 > 1$. $\square$

The following two lemmas are evident.

Lemma 2. *The equality* $\gamma_s'(Q_1) = 1$ *holds. The corresponding* SEDF *has the property* VS1.

Lemma 3. *The equality* $\gamma_{st}'(Q_2) = 1$ *holds. The corresponding STEDF has the property VS2.*

Remark. The cube graph Q_1 satisfies $Q_1 \cong K_2$ and no STEDF exists in it.

Lemma 4. *Let* f *be* SEDF *of* Q_n *having the property* VS1. *Then there exists a* SEDF $\widehat{f}$ *of* Q_{nt2} *having the property* VS1 *and* $w(\widehat{f}) = 4w(f)$.

Proof. For any i, j from $\{0,1\}$ let $V(i,j)$ denote the set of all Boolean vectors of dimension $n+2$ whose $(n+1)$-st coordinates is i and whose $(n+2)$-nd coordinate is j. Let $G(i,j)$ be the subgraph of Q_{n+2} induced by $V(i,j)$. Evidently $G(i,j) \cong Q_n$. There exists an isomorphism φ_{ij} of $G(i,j)$ onto Q_n such that the image of $(v_1,\ldots,v_n,i,j)$ in φ_{ij} is $(v_1,\ldots,v_n)$. Let the function f be given on Q_n. For each e belonging to some $G(i,j)$ we put $\widehat{f}(e) = f(\varphi_{ij}(e))$. If e joins a vertex of $G(0,0)$ with a vertex of $G(0,1)$ or a vertex of $G(1,0)$ with a vertex of $G(1,1)$, then $\widehat{f}(e) = -1$. If e joins a vertex of $G(0,0)$ with a vertex of $G(1,0)$ or a vertex of $G(0,1)$ with a vertex of $G(1,1)$, then $f(e) = 1$. The restriction of f onto $G(i,j)$ for any i,j has the property VS1; this follows from the construction. In Q_{n+2} each vertex v of $V(i,j)$ is incident with two further edges, one of which has the value 1, the order -1, therefore the sum of values of incident edges is not changed. Hence $\widetilde{f}$ has VS1 and it is a SEDF on Q_{n+2}. Evidently $w(\widehat{f}) = 4w(f)$, because there are four graphs $G(0,0)$, $G(0,1)$, $G(1,0)$, $G(1,1)$. $\square$

Lemma 5. *Let* f *be a* STEDF *of* Q_n *having the property* VS2. *Then there exists a* STEDF $\widetilde{f}$ *of* Q_{n+2} *having the property* VS2 and $w(\widehat{f}) = 4w(f)$.

Proof is analogous. $\square$

Lemma 6. *Let* f *be a* SEDF *on* Q_n *having the property* VS1. *Then there exists a* SEDF $\widetilde{f}$ *of* Q_{n+2} *such that* $w(\widetilde{f}) = 2w(f)$.

P r o o f. For each $i \in \{0,1\}$ let $V(i)$ be the set of all Boolean vectors of dimension $n+1$ which have the $(n+1)$-st coordinate equal to i. Let $G(i)$ be the subgraph of Q_{n+1} induced by $V(i)$. There exists an isomorphism ψ_i of $G(i)$ onto Q_n such that the image of $(v_1,\ldots,v_n,i)$ in ψ_i is $(v_1,\ldots,v_n)$. Let the function f be given on Q_n. For each e belonging to $G(i)$ for $i \in \{0,1\}$ we put $\widetilde{f}(e) = f(\psi_i(e))$. In Q_{n+1} we may consider the sets M^0_{n+1} and M^1_{n+1}. We put $\widetilde{f}(e) = -1$ for $e \in M^0_{n+1}$ and $f(e) = 1$ for $e \in M^1_{n+1}$. If e is an edge joining vertices u, v of $V(i)$ for some $i \subset \{0,1\}$, then in $G(i)$ we have $s(G(i), f, u) \geqslant 1$, $s(G(i), f, v) \geqslant 1$. Without loss of generality we may suppose that in Q_{n+1} the vertex u is incident with an edge from M^0_{n+1} and the vertex v is incident with an edge of M^2_{n+1}. Thus $s(Q_{n+1}, \widetilde{f}, u) = s(Q(i), f, u) - 1$, $s(Q_{n+1}, f, v) = s(G(i), \widetilde{f}, v) + 1$ and $f(N[e]) = s(Q_{n+1}, \widetilde{f}, u) + s(Q_{n+1}, \widetilde{f}, v) - \widetilde{f}(e) \geqslant 0 + 2 - 1 = 1$. If e is an edge of Q_{n+1} and $e \in M_{n+1}$ and e joins a vertex u of $G(0)$ with a vertex v of $G(1)$, then $f(N[e]) = s(G(0), \widetilde{f}, u) + s(G(1), \widetilde{f}, v) + \widetilde{f}(e) \geqslant 1 + 1 - 1 = 1$. Therefore $\widetilde{f}$ is a SEDF and evidently $w(\widetilde{f}) = 2w(f)$. $\square$

Lemma 7. *Let f be a* STEDF *on Q_n having the property* VS2. *Then there exists a* STEDF *$\widetilde{f}$ on Q_{n+1} such that $w(\widetilde{f}) = 2w(f)$.*

P r o o f is analogous. $\square$

Theorem 2. *For each positive integer n the following inequality holds:*

$$\gamma'_s(Q_n) \leqslant 2^{n-1}.$$

P r o o f. For all odd positive integers we prove the assertion by induction using Lemma 2 and Lemma 4. Then we prove it for even positive integers n using Lemma 6. $\square$

Theorem 3. *For each integer $n \geqslant 2$ the following inequality holds:*

$$\gamma'_{st}(Q_n) \leqslant 2^n.$$

P r o o f is analogous. $\square$

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